J. Fluid Mech. (1990), *vol.* 211, *pp.* 333–353
Printed in Great Britain

The hydrodynamic force on an oscillating ship with low forward speed

By G. X. WU AND R. EATOCK TAYLOR†
Department of Mechanical Engineering, University College London, Torrington Place, London WC1E 7JE, UK

(Received 9 September 1987 and in revised form 18 January 1989)

The mathematical formulation of the linearized potential theory for a slowly translating body undergoing oscillations in infinitely deep water is derived based on a perturbation series in terms of forward speed. It is shown that the hydrodynamic force on the oscillating body can be obtained from the solution of the velocity potential without forward speed, if second-order terms in forward speed are neglected. An example of a submerged circular cylinder is discussed. The numerical results are compared with the general solution of the linearized potential theory by a coupled finite-element method (Wu & Eatock Taylor 1987) which is not restricted to low forward speeds. Very good agreement is found. The nonlinear effect of the steady potential on the hydrodynamic forces is also discussed and is illustrated for a floating semicircular cylinder.

1. Introduction

In the prediction of motions of marine vehicles advancing in waves, a common method is to use strip theory. Based on the assumptions of linearized potential flow analysis, the strip theory further assumes that the fluid flow corresponding to each section of the body is two dimensional. The application of this theory requires the body to be slender and the relative magnitudes of forward speed and encounter frequency to be limited to an appropriate range of values. More recently there have been attempts, initiated by Chang (1977), to obtain the solution of the linear velocity potential using three-dimensional methods. It has been observed that the three-dimensional theory generally gives better agreement with experimental data on ship models, but it requires much more effort. The numerical calculation is extremely expensive compared with strip theory.

To reduce the computer time, it is necessary to use an appropriate numerical procedure and Green function form. One approach is to exploit any simplifications made possible by considering the case of low forward speed at which many merchant ships operate. Huijsmans (1986) expanded the source distribution over the body surface in a perturbation series of forward speed and neglected the terms of order $O(U^2)$. He was then able to use a correspondingly simplified Green function form. However, it may be noticed that this Green function has a second-order singularity which may not be easy to deal with.

In this work, we analyse the problem of a body advancing in waves at low forward speed in a different manner. Instead of expanding the source distribution, we use the perturbation series of the potential in terms of forward speed and neglect terms of

† Present address: Department of Engineering Science, University of Oxford, Parks Road, Oxford OX1 3PJ, UK.

order $O(U^2)$ or higher. We then show that to calculate the hydrodynamic force on the advancing body we only need the solution without forward speed. Equations for the forces are obtained which contain an integral over the free surface of the potential without forward speed.

In addition to simplifying the problem, the present formulation can also provide useful insight into the low-forward-speed behaviour of more general analyses, intended for any value of forward speed. Such formulations are under development by a number of investigators, and linear three-dimensional theory is being used to obtain results for both submerged bodies and slender ships. In the general formulation, the Green function with forward speed depends on four wavenumbers. As the forward speed becomes very small, two of these wavenumbers become very large, causing a highly oscillatory behaviour of the Green function and substantial computational difficulties. The present formulation, however, is always stable and the numerical solution should therefore tend to the correct limit as the forward speed tends to zero.

Another difficulty arises when one attempts to develop the general formulation for an arbitrary body with forward speed at the free surface. Unless the body is sufficiently slender (or submerged), the assumptions of the fully linear theory can no longer be justified, as discussed by Tuck (1965). An attempt has therefore been made here to investigate the nonlinear effect of the steady potential at the free surface on the hydrodynamic force. We have derived a mathematical model which takes account of this nonlinear coupling between steady and unsteady components of the velocity potential, but retains the linearization of the unsteady potential associated with small oscillations of the body. It is found that this coupling effect has important implications for satisfaction of the reverse flow relations of Timman & Newman (1962).

A possible criticism of this and many other models is that no attempt is made to account for flow separation. Our analysis is based on the assumptions of potential flow theory, which have been very widely adopted in investigations of ship motions. But these clearly impose certain limitations on the applicability of the theory. For a body in a real fluid, the effect of viscosity is known to form a thin boundary layer, which will remain thin and attached provided that the body is streamlined: outside this boundary layer, the assumptions of potential flow theory are valid. It has also been found that potential flow predictions of the motions of bluff bodies in waves without forward speed generally agree well with experimental data, except in circumstances where separation is induced by sharp corners and bilge keels (e.g. in barge rolling). It seems reasonable to proceed on the basis that the addition of a small forward speed is unlikely to negate the applicability of potential flow theory for a slender body such as a ship in waves. The present paper adopts this approach. It aims to produce semi-analytical results, which may be used to assist the development of a general three-dimensional formulation for arbitrary (but slender) ship forms. The solutions obtained here are limited to circular and semicircular cylinders moving transverse to their axes; and these are clearly not slender or streamlined bodies. But the case for using potential flow theory here rests on the need to provide results which can validate the numerical formulations required for arbitrary ship forms. The results also shed some light on the implications of certain terms in the potential flow formulation.

The paper is organized as follows. Section 2 introduces the general boundary-value problem, and discusses aspects of the linearization. The implications of the low-forward-speed assumption for a fully linear analysis are then investigated in §3. This analysis is appropriate to a slender body at the free surface, or a deeply submerged body. General expressions are obtained for the added mass and damping of two- and three-dimensional bodies at low forward speed. Based on the analytical solution for an oscillating submerged circular cylinder without forward speed, given by Ogilvie (1963), results are obtained showing the influence of forward speed on the hydrodynamic coefficients. Comparison is made with the general numerical solution of the linearized potential problem using the coupled finite element method (Wu & Eatock Taylor 1987).

The nonlinear effect of the steady potential at the free surface is examined in §4. This leads to additional contributions to the expressions for the hydrodynamic coefficients. Numerical results are obtained for a floating semicircular cylinder, based on the method of multipole expansions given by Ursell (1949). In §5 some conclusions are drawn concerning the reverse flow relationship, and the influence of the nonlinear effect.

2. The boundary-value problem

We define the right-handed coordinate system $O-xyz$ so that x points in the direction of forward speed U and z upwards. The origin of the coordinates is located on the undisturbed free surface, and moves with the body at the same forward speed. We consider the case of forced periodical motions at the encounter frequency ω and take the time factor as $e^{i\omega t}$. Then the total potential can be written as a linear superposition

$$\Phi = -Ux + U\bar{\phi} + \sum_{j=1}^{6} \eta_j \phi_j e^{i\omega t}, \tag{1}$$

where $\bar{\phi}$ is the potential due to unit forward speed, ϕ_j is the radiation potential corresponding to each of the six rigid body degrees of freedom of the body and η_j is the corresponding motion amplitude which is assumed to be small throughout this paper. Based on the usual assumption of potential flow theory we have

$$\nabla^2 \Phi = 0, \tag{2}$$

in the whole fluid domain, and in deep water Φ satisfies

$$\lim_{z \to -\infty} \nabla \Phi = 0. \tag{3}$$

On the body surface S_0, the steady potential satisfies

$$\frac{\partial \bar{\phi}}{\partial n} = n_x, \tag{4}$$

where n_x is the component in the x-direction of the inward normal $\boldsymbol{n}$ on S_0. The components of the radiation potential satisfy (Newman 1978)

$$\frac{\partial \phi_j}{\partial n} = i\omega n_j + U m_j, \tag{5}$$

on the body surface, where

$$(n_1, n_2, n_3) = (n_x, n_y, n_z), \tag{6a}$$

$$(n_4, n_5, n_6) = \boldsymbol{X} \times \boldsymbol{n}, \tag{6b}$$

$$U(m_1, m_2, m_3) = -(\boldsymbol{n} \cdot \nabla)\, \boldsymbol{W}, \tag{6c}$$

$$U(m_4, m_5, m_6) = -(\boldsymbol{n} \cdot \nabla)(\boldsymbol{X} \times \boldsymbol{W}), \tag{6d}$$

$$\boldsymbol{W} = U \nabla(\bar{\phi} - x), \tag{6e}$$

and $\boldsymbol{X}$ is the position vector of a point on S_0 relative to the centre of mass of the body.

The boundary conditions on the free surface S_F are, of course, highly nonlinear because of the steady potential, even in the case of small oscillatory motions. We start from the general free-surface conditions derived by Newman (1978). For the steady potential we have

$$\tfrac{1}{2}\boldsymbol{W}\cdot\nabla(W^2)+gU\bar{\phi}_z=0, \tag{7a}$$

on $z=\bar{\zeta}$, where

$$\bar{\zeta}=-\frac{1}{2g}(W^2-U^2)_{z=\bar{\zeta}}; \tag{7b}$$

and for the unsteady body motion potential

$$\tfrac{1}{2}\boldsymbol{W}\cdot\nabla(W^2)+gU\bar{\phi}_z-\omega^2\phi+2\mathrm{i}\omega\boldsymbol{W}\cdot\nabla\phi+\boldsymbol{W}\cdot\nabla(\boldsymbol{W}\cdot\nabla\phi)+\tfrac{1}{2}\nabla\phi\cdot\nabla(W^2)+g\phi_z=0, \tag{8}$$

on $z=\zeta$, where ζ is the total free-surface elevation and

$$\phi=\sum_{j=1}^{6}\eta_j\phi_j.$$

Equation (8) has retained the nonlinear terms of $\bar{\phi}$, but neglected nonlinear terms in ϕ_j on the basis of the small-motion assumption and the linearization implicit in (1).

In general the solution for the steady potential would require an iterative procedure, because of the nonlinear coupling between (7a) and (7b). If however we assume that $U/(gl)^{\frac{1}{2}}$ is small, where l is a length characterizing the variation of $\bar{\phi}$, then the free-surface condition on the steady potential can be taken as

$$\bar{\phi}_z=0 \quad \text{on } z=0=\bar{\zeta}, \tag{9}$$

after terms of order $O(U^2)$ or higher are neglected. Applying the same assumption to the free-surface condition for the radiation potentials, we obtain

$$gU\bar{\phi}_z-\omega^2\phi+2\mathrm{i}\omega\boldsymbol{W}\cdot\nabla\phi+g\phi_z=0, \tag{10}$$

on $z=\zeta$, where $\zeta=-(\mathrm{i}\omega/g)\phi-(1/g)\,\boldsymbol{W}\cdot\nabla\phi$ since $\bar{\zeta}=0$. Expanding the last equation into a Taylor series about $z=0$, using

$$\begin{aligned}\bar{\phi}_z|_{z=\zeta}&=[\bar{\phi}_z+\zeta\bar{\phi}_{zz}]_{z=0}+O(\zeta^2)\\&=-\frac{\mathrm{i}\omega}{g}\phi\bar{\phi}_{zz}-\frac{1}{g}\boldsymbol{W}\cdot\nabla\phi\bar{\phi}_{zz}+O(\zeta^2),\end{aligned} \tag{11}$$

and noticing that the expansions of the other terms are of higher order in ϕ we have

$$-\mathrm{i}\omega\phi\bar{\phi}_{zz}-\omega^2\phi+2\mathrm{i}\omega\boldsymbol{W}\cdot\nabla\phi+g\phi_z=0, \tag{12}$$

on $z=0$. This has been obtained by linearizing with respect to the body motion potential ϕ, but retaining the product term involving the steady potential. In a fully linear theory, this term is absent from the free-surface boundary condition.

Since (12) is linear with respect to ϕ, it is also satisfied by the component ϕ_j. Provided that these boundary-value problems can be solved satisfactorily, the added masses μ_{ij} and damping coefficients λ_{ij} can then be obtained from (Newman 1978)

$$\begin{aligned}\tau_{ij}&=\omega^2\mu_{ij}-\mathrm{i}\omega\lambda_{ij}\\&=-\rho\int_{S_0}[\mathrm{i}\omega\phi_j+\boldsymbol{W}\cdot\nabla\phi_j]\,n_i\,\mathrm{d}S,\end{aligned} \tag{13}$$

where ρ is the density of the fluid.

3. A fully linear analysis for low forward speed

3.1. *A perturbation expansion for the velocity potential* ϕ_j

For a slender body at the free surface, or for a deeply submerged body, it is often assumed that the disturbance at the free surface due to steady forward speed is small and its product terms and those with the oscillating potential may be neglected in the free-surface boundary condition. The latter may then be expressed in the form (Newman 1978)

$$\mu\bar{\phi}_z + \bar{\phi}_{xx} = 0, \tag{14a}$$

$$\phi_{jz} + \frac{\tau^2}{\nu}\phi_{jxx} - 2\mathrm{i}\tau\phi_{jx} - \nu\phi_j = 0, \tag{14b}$$

on the undisturbed free surface S_F, where $\mu = g/U^2$, $\tau = \omega U/g$ and $\nu = \omega^2/g$. For a consistent analysis of a body at the free surface, the influence of the steady potential on the body-surface condition expressed by (5) should also be neglected. For a deeply submerged body, however, the steady potential may be negligible on the free surface but not on the body: in this case the complete expression given in (5) may be more appropriate.

To complete the specification of the boundary-value problem, we need to include radiation conditions on a surface S_∞ at infinity. For the steady potential this is usually represented by the assumptions that there is no wave due to $\bar{\phi}$ far in front of the body, but there are waves far behind the body. The assumed radiation condition for ϕ_j states that the wave whose group velocity is larger than the forward speed is in front of the body; otherwise the waves propagate behind.

Even after linearization, the form of the free-surface condition in (14*b*) causes considerable difficulties: the influence of forward speed U on solution for ϕ_j is particularly complex. Grue & Palm (1985) considered a two-dimensional problem of a submerged circular cylinder. For this particular case they were able to write the source distribution over the cylinder surface in a Fourier series, and an analytical solution was thereby obtained. The linear solution for an arbitrary cylinder submerged below the surface was obtained by Wu & Eatock Taylor (1987) using a coupled finite-element method. A numerical solution of the three-dimensional linear problem was first obtained by Chang (1977), by distributing sources over the body surface and waterline. Other linear formulations employing distributions of sources have been given by Inglis & Price (1981) and Guevel & Bougis (1982).

Here we make use of the assumption of low forward speed to derive a simpler formulation. We write the radiation potential as a perturbation expansion in $u = U/\omega a$ and neglect terms of $O(u^2)$ and higher. Thus

$$\phi_j = \mathrm{i}\omega\phi_j^0 + \frac{U}{a}\phi_j^1, \tag{15}$$

where a may be defined as the typical dimension of the body. Substituting (15) into the governing equations for ϕ_j and rearranging results according to the order of U, we obtain from the zeroth-order terms

$$\phi_{jz}^0 - \nu\phi_j^0 = 0 \quad (S_F) \tag{16}$$

$$\frac{\partial\phi_j^0}{\partial n} = n_j \quad (S_0); \tag{17}$$

and from the first-order terms

$$\phi^1_{jz} - \nu\phi^1_j = -2\nu a \phi^0_{jx} \quad (S_F) \tag{18}$$

$$\frac{\partial \phi^1_j}{\partial n} = a m_j \quad (S_0). \tag{19}$$

Here ϕ^0_j are the potentials without forward speed, whose solution is much easier to obtain than that with forward speed. ϕ^1_j is related to the derivative of ϕ_j with respect forward speed at $U = 0$, and its solution would need additional effort. However, we will show that to obtain τ_{ij} in (13), we only need the solution of ϕ^0_j.

3.2. *Expressions for hydrodynamic coefficients in terms of* ϕ^0_j

Ogilvie & Tuck (1969) have shown that

$$\int_{S_0} (\boldsymbol{W} \cdot \nabla \phi_j)\, n_i \,\mathrm{d}S = -U \int_{S_0} \phi_j m_i \,\mathrm{d}S - U \int_{C_0} \phi_j \bar{\phi}_z n_i \,\mathrm{d}l, \tag{20}$$

where C_0 is the waterline of the body. At low forward speed, because of the free-surface condition (9) for $\bar{\phi}$, equation (20) becomes

$$\int_{S_0} (\boldsymbol{W} \cdot \nabla \phi)\, n_i \,\mathrm{d}S = -U \int_{S_0} \phi_j m_i \,\mathrm{d}S. \tag{21}$$

Thus the hydrodynamic coefficients in (13) may be written

$$\begin{aligned}
\tau_{ij} &= -\rho \int_{S_0} [\mathrm{i}\omega \phi_j n_i - U m_i \phi_j] \,\mathrm{d}S \\
&= -\rho \int_{S_0} [\mathrm{i}\omega n_i - U m_i] \left[\mathrm{i}\omega \phi^0_j + \frac{U}{a}\phi^1_j\right] \mathrm{d}S \\
&= -\rho \int_{S_0} \left[-\omega^2 \phi^0_j n_i + \mathrm{i}\omega U \left(\frac{\phi^1_j n_i}{a} - m_i \phi^0_j\right)\right] \mathrm{d}S \\
&= \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho \mathrm{i}\omega U \int_{S_0} m_i \phi^0_j \,\mathrm{d}S - \rho \mathrm{i}\omega \frac{U}{a} \int_{S_0} \phi^1_j n_i \,\mathrm{d}S,
\end{aligned} \tag{22}$$

where a_{ij} and b_{ij} are the added masses and damping coefficients without forward speed. We have neglected the term of order U^2 in the derivation. Equation (22) can be easily computed from ϕ^0_j except the last term. However, we can write

$$\int_{S_0} \phi^1_j n_i \,\mathrm{d}S = \int_{S_0} \phi^1_j \frac{\partial \phi^0_i}{\partial n} \mathrm{d}S = \int_{S_0} \phi^0_i \frac{\partial \phi^1_j}{\partial n} \mathrm{d}S - \int_S \left[\phi^1_j \frac{\partial \phi^0_i}{\partial n} - \phi^0_i \frac{\partial \phi^1_j}{\partial n}\right] \mathrm{d}S,$$

where $S = S_F + S_\infty$. Invoking the boundary conditions on S_0 and S_F, we obtain

$$\begin{aligned}
\int_{S_0} \phi^1_j n_i \,\mathrm{d}S &= a \int_{S_0} \phi^0_i m_j \,\mathrm{d}S - \int_{S_F} [\phi^1_j \nu \phi^0_i - \phi^0_i(\nu\phi^1_j - 2\nu a \phi^0_{jx})] \,\mathrm{d}S \\
&\quad - \int_{S_\infty} \left[\phi^1_j \frac{\partial \phi^0_i}{\partial n} - \phi^0_i \frac{\partial \phi^1_j}{\partial n}\right] \mathrm{d}S \\
&= a \int_{S_0} \phi^0_i m_j \,\mathrm{d}S - 2\nu a \int_{S_F} \phi^0_i \phi^0_{jx} \,\mathrm{d}S - \int_{S_\infty} \left[\phi^1_j \frac{\partial \phi^0_i}{\partial n} - \phi^0_i \frac{\partial \phi^1_j}{\partial n}\right] \mathrm{d}S.
\end{aligned} \tag{23}$$

Equation (23) still involves the potential ϕ^1_j in the integral at infinity. This may be rewritten, using the appropriate radiation condition. For this purpose we employ the far-field asymptotic form of the Green function, for a wave source at small forward speed. For a three-dimensional source located at $(0, 0, \zeta)$ in infinite water depth, the Green function may be written (see Haskind 1946):

$$G \approx 2(1+2\tau\cos\theta)\left(\frac{2\pi\nu}{R}\right)^{\frac{1}{2}}\exp\left[\nu(1+2\tau\cos\theta)(z+\zeta-\mathrm{i}R)-\tfrac{1}{4}\mathrm{i}\pi\right], \tag{24}$$

as $R \to \infty$, where

$$x = R\cos\theta, \quad y = R\sin\theta.$$

This is based on the assumption that $\tau < 0.25$. From this equation we find

$$\frac{\partial G}{\partial R} = -\mathrm{i}\nu(1+2\tau\cos\theta)\,G \quad (R\to\infty). \tag{25}$$

It follows that the radiation condition on ϕ_j at small forward speed may be written as

$$\frac{\partial\phi_j}{\partial R} = -\mathrm{i}\nu(1+2\tau\cos\theta)\,\phi_j. \tag{26}$$

Using (15), we obtain

$$\frac{\partial\phi_j^0}{\partial R} = -\mathrm{i}\nu\phi_j^0, \tag{27a}$$

which is the well-known radiation condition without forward speed, and

$$\frac{\partial\phi_j^1}{\partial R} = -\mathrm{i}\nu\phi_j^1 + 2\nu^2 a\cos\theta\phi_j^0. \tag{27b}$$

A possible objection to the present formulation is raised by (27*b*), which implies that

$$\phi_j^1 \to 2\nu^2 aR\cos\theta\phi_j^0 \quad (R\to\infty).$$

This suggests that the proposed expansion of the radiation potential in terms of a forward speed perturbation parameter is only valid when $R < \infty$. In particular, for a given forward speed the accuracy of the potential given by (15) decreases as R increases. The following comments, however, appear pertinent. We are here concerned only with the hydrodynamic forces, as given by (22). It therefore seems reasonable to proceed on the basis that R is fixed and U is sufficiently small to ensure satisfactory behaviour of the expansion. We may then compare the results obtained in this way with those derived from the more general numerical procedure, which does not require the perturbation expansion for ϕ_j.

On this basis, we substitute (27) into (23) to obtain

$$\frac{1}{a}\int_{S_0}\phi_j^1 n_i\,\mathrm{d}S = \int_{S_0}\phi_i^0 m_j\,\mathrm{d}S - 2\nu\int_{S_F}\phi_i^0\phi_{jx}^0\,\mathrm{d}S + 2\nu^2\int_{S_\infty}\phi_i^0\phi_j^0\cos\theta\,\mathrm{d}S. \tag{28}$$

Thus for the three-dimensional problem

$$\tau_{ij} = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho\mathrm{i}\omega U\int_{S_0}[m_i\phi_j^0 - m_j\phi_i^0]\,\mathrm{d}S + \rho\mathrm{i}\omega U\left[2\nu\int_{S_F}\phi_i^0\phi_{jx}^0\,\mathrm{d}S - 2\nu^2\int_{S_\infty}\phi_i^0\phi_j^0\cos\theta\,\mathrm{d}S\right]. \tag{29}$$

For the two-dimensional problem, the radiation conditions on ϕ_j^0 and ϕ_j^1 at small forward speed may be written as

$$\frac{\partial \phi_j^0}{\partial x} = -\mathrm{i}\nu\phi_j^0, \quad \frac{\partial \phi_j^1}{\partial x} = -\mathrm{i}\nu\phi_j^1 + 2\nu^2 a\phi_j^0, \tag{30a, b}$$

when $x \to +\infty$, and

$$\frac{\partial \phi_j^0}{\partial x} = \mathrm{i}\nu\phi_j^0, \quad \frac{\partial \phi_j^1}{\partial x} = \mathrm{i}\nu\phi_j^1 + 2\nu^2 a\phi_j^0, \tag{31a, b}$$

when $x \to -\infty$. Thus, the added mass and damping coefficients can be obtained from

$$\tau_{ij} = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho\mathrm{i}\omega U \int_{S_0} (m_i \phi_j^0 - m_j \phi_i^0)\,\mathrm{d}S + 2\nu\rho\mathrm{i}\omega U\left[\int_{S_F} \phi_i^0 \phi_{jx}^0 \,\mathrm{d}x + \nu\left(\int_{S_{-\infty}} \phi_i^0 \phi_j^0 \,\mathrm{d}z - \int_{S_{+\infty}} \phi_i^0 \phi_j^0 \,\mathrm{d}z\right)\right]. \tag{32}$$

Care is needed in calculation of the integrals in (29) and (32), since the integrations over S_F or S_∞ do not converge; but it is easy to show that in both cases the sum of these two integrals is convergent.

Alternatively, using the asymptotical behaviour of ϕ_j^0 at infinity (e.g. Mei 1982), we may write the last two integrals in (29) as

$$I = 2\nu \int_{S_F} \phi_i^0 \phi_{jx}^0 \,\mathrm{d}S - \nu \int_{C_\infty} \phi_i^0 \phi_j^0 \cos\theta \,\mathrm{d}l,$$

where C_∞ is the waterline at infinity. We use Stokes theorem for the line integral, and retain at this stage the possibility of a body piercing the free surface (since we shall be using a similar transformation in a later section). We therefore obtain

$$\begin{aligned} I &= 2\nu \int_{S_\infty} \phi_i^0 \phi_{jx}^0 \,\mathrm{d}S - \nu \int_{S_F} \frac{\partial}{\partial x}(\phi_i^0 \phi_j^0)\,\mathrm{d}S - \nu \int_{C_0} \phi_i^0 \phi_j^0 \,\mathrm{d}y \\ &= \nu \int_{S_F} (\phi_i^0 \phi_{jx}^0 - \phi_{ix}^0 \phi_j^0)\,\mathrm{d}S - \nu \int_{C_0} \phi_i^0 \phi_j^0 \,\mathrm{d}y, \end{aligned}$$

where C_0 would be the waterline of the body. For a submerged body the last term is of course absent. Thus, (29) becomes

$$\tau_{ij} = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho\mathrm{i}\omega U \int_{S_0} (m_i \phi_j^0 - m_j \phi_i^0)\,\mathrm{d}S + \rho\mathrm{i}\omega U\nu\left[\int_{S_F} (\phi_i^0 \phi_{jx}^0 - \phi_{ix}^0 \phi_j^0)\,\mathrm{d}S - \int_{C_0} \phi_i^0 \phi_j^0 \,\mathrm{d}y\right]. \tag{33}$$

Similarly in two dimensions, (32) can be written as

$$\tau_{ij} = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho\mathrm{i}\omega U \int_{S_0} (m_i \phi_j^0 - m_j \phi_i^0)\,\mathrm{d}S + \rho\mathrm{i}\omega U\nu\left[\int_{S_F} (\phi_i^0 \phi_{jx}^0 - \phi_{ix}^0 \phi_j^0)\,\mathrm{d}S - (\phi_j^0 \phi_i^0)_P + (\phi_j^0 \phi_i^0)_Q\right] \tag{34}$$

where P and Q would be upstream and downstream points of intersection of the body with the free surface.

We have therefore obtained in (33) and (34) – corresponding to three dimensions and two dimensions respectively – expressions for the speed-dependent hydrodynamic coefficients, which only involve the velocity potential ϕ_j^0 for the zero-forward-speed problem. The terms proportional to forward speed include integrals over the free surface and a contribution from the intersection of the free surface and the body. For submerged bodies, such as the circular cylinder considered in §3.4, the last contribution is of course absent.

3.3. *Properties of the hydrodynamic coefficients*

It is of interest to consider certain general features of the resulting expressions. First we note the form of the diagonal terms τ_{jj}, which in three dimensions reduce to

$$\tau_{jj} = \omega^2 a_{jj} - \mathrm{i}\omega b_{jj} - \rho \mathrm{i}\omega U \nu \int_{C_0} (\phi_j^0)^2 \,\mathrm{d}y. \tag{35}$$

For the submerged body the waterline integral vanishes, and we conclude that forward speed does not affect the hydrodynamic coefficients τ_{jj} of submerged bodies at least until the second order. If the linear theory were applied to the case of a floating body, the line integral in (35) would not in general vanish; but if the body were symmetrical about its middle section $x = 0$, the line integral would also equal zero. Similar conclusions hold for the two-dimensional case.

From (33), we can derive the Timman–Newman relation (Timman & Newman 1962; Newman 1965). By reversing the direction of forward speed, we obtain

$$\tau_{ij}(-U) = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} - \rho \mathrm{i}\omega U \int_{S_0} [m_i \phi_j^0 - m_j \phi_i^0] \,\mathrm{d}S$$
$$- \rho \mathrm{i}\omega U \nu \left[\int_{S_F} (\phi_i^0 \phi_{jx}^0 - \phi_{ix}^0 \phi_j^0) \,\mathrm{d}S - \int_{C_0} \phi_i^0 \phi_j^0 \,\mathrm{d}y \right].$$

Following the argument of Timman & Newman (1962) that the line integral in this equation is negligible for slender bodies and disappears for submerged bodies, and noticing the well-known relation $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$ (e.g. Mei 1982), we obtain the expected result

$$\tau_{ij}(-U) = \tau_{ji}(U). \tag{36}$$

It should be noted, however, that if the linear theory is applied to an arbitrary body at the free surface, this relationship is not exactly satisfied. It is subject to the assumptions of low forward speed and negligible waterline integral. An illustration that for large forward speed (36) may not be satisfied has been given by Wu & Eatock Taylor (1988).

An application of (36) is when the body has fore and aft symmetry. Careful analysis gives (Timman & Newman 1962)

$$\tau_{ij} = -\tau_{ji} \quad (i \neq j), \tag{37a}$$

except

$$\tau_{15} = \tau_{51}, \quad \tau_{24} = \tau_{42}. \tag{37b, c}$$

3.4. *Results for a submerged circular cylinder*

The foregoing theory is now applied to the case of a submerged body at low forward speed. We recall that the analysis is based on linearized potential theory, and in particular the free-surface boundary condition has been simplified in several respects. First, the effect of the steady potential is assumed to be small at the free surface. This assumption is similar to that adopted by Grue & Palm (1985) and Wu & Eatock Taylor (1987). Secondly, we have made the further assumption that since $U/(gl)^{\frac{1}{2}} \ll 1$, the steady potential satisfies the rigid-surface condition (9) and all terms in U^2 are omitted from the free-surface condition on the body motion potential. We have used the perturbation procedure to write the radiation potential as the sum of two terms, one independent of U and one term linearly proportional to U. This is based on $U/\omega a$ being small, or $Fn = U/(ga)^{\frac{1}{2}} \ll (\nu a)^{\frac{1}{2}}$. Finally we have assumed $U\omega/g < 0.25$, in order that the desired behaviour at infinity is achieved.

We consider the problem of an oscillating circular cylinder of radius a in infinite water depth, such that the distance from its centre to the free surface is h (with $h > a$). The solution without forward speed is well known. Ogilvie (1963) for example has given a detailed derivation, using the well-known multipole expansion method (Thorne 1953). Following this formulation, we write the heave potential in the form

$$\phi_3^0 = \sum_{m=1}^{\infty} p_m \left[\frac{a^m \cos m\theta}{r^m} + \frac{a^m}{(m-1)!} \oint_0^{\infty} k^{m-1} \frac{k+\nu}{k-\nu} \mathrm{e}^{k(z-h)} \cos kx \,\mathrm{d}k\right], \tag{38a}$$

and the sway potential

$$\phi_1^0 = \sum_{m=1}^{\infty} q_m \left[\frac{a^m \sin m\theta}{r^m} + \frac{a^m}{(m-1)!} \oint_0^{\infty} k^{m-1} \frac{k+\nu}{k-\nu} \mathrm{e}^{k(z-h)} \sin kx \,\mathrm{d}k\right], \tag{38b}$$

where the polar coordinate system (r, θ) is defined by

$$x = r\sin\theta, \quad z+h = r\cos\theta. \tag{39a, b}$$

Using the relations

$$\mathrm{e}^{k(z+h)} \cos kx = \sum_{m=0}^{\infty} \frac{(kr)^m \cos m\theta}{m!}, \tag{40a}$$

$$\mathrm{e}^{k(z+h)} \sin kx = \sum_{m=1}^{\infty} \frac{(kr)^m \sin m\theta}{m!}, \tag{40b}$$

we can write (38) as

$$\phi_3^0 = \sum_{m=0}^{\infty} p_m \frac{a^m \cos m\theta}{r^m} + \sum_{m=0}^{\infty} \frac{r^m}{m!} \cos m\theta \sum_{n=1}^{\infty} p_n \frac{a^n}{(n-1)!} I_{n+m-1}, \tag{41a}$$

$$\phi_1^0 = \sum_{m=1}^{\infty} q_m \frac{a^m \sin m\theta}{r^m} + \sum_{m=1}^{\infty} \frac{r^m}{m!} \sin m\theta \sum_{n=1}^{\infty} q_n \frac{a^n}{(n-1)!} I_{n+m-1}, \tag{41b}$$

where

$$\begin{aligned} I_{m-1} &= \oint_0^{\infty} k^{m-1} \frac{k+\nu}{k-\nu} \mathrm{e}^{-2kh} \,\mathrm{d}k \\ &= \frac{(m-1)!}{(2h)^m} + 2\nu \left\{\frac{(m-2)!}{(2h)^{m-1}} + \ldots \frac{\nu^{m-2}}{2h} - \nu^{m-1} \mathrm{e}^{-2\nu h} [\mathrm{Ei}(2\nu h) + \pi \mathrm{i}]\right\}, \end{aligned} \tag{42}$$

and $\mathrm{Ei}(x)$ is the exponential integral (Abramowitz & Stegun 1965). By imposing the body surface condition

$$\frac{\partial \phi_3^0}{\partial r} = \cos\theta, \quad \frac{\partial \phi_1^0}{\partial r} = \sin\theta, \tag{43a, b}$$

we obtain the linear equations for p_m and q_m

$$-\frac{m}{a} p_m + \sum_{n=1}^{\infty} \frac{a^{m+n-1} I_{m+n-1}}{(m-1)!(n-1)!} p_n = \delta_{m1}, \tag{44a}$$

$$-\frac{m}{a} q_m + \sum_{n=1}^{\infty} \frac{a^{m+n-1} I_{m+n-1}}{(m-1)!(n-1)!} q_n = \delta_{m1}, \tag{44b}$$

where δ_{m1} is the Kronecker delta function. Comparison of (44*a*) and (44*b*) immediately shows that $p_m = q_m$. The solution of this equation may be obtained by truncating the series at a finite number of terms, sufficient to yield the desired accuracy. From (41) and (44) we then write ϕ_3^0 and ϕ_1^0 on the body surface as

$$\phi_3^0 = \sum_{m=1}^{\infty}\left[2p_m + \frac{a}{m}\delta_{m1}\right]\cos m\theta, \tag{45a}$$

$$\phi_1^0 = \sum_{m=1}^{\infty}\left[2q_m + \frac{a}{m}\delta_{m1}\right]\sin m\theta. \tag{45b}$$

To calculate the hydrodynamic coefficients in (34), we also need to find the steady potential $\bar{\phi}$. This can be obtained most simply by taking $\nu = 0$ in (41*b*). We have

$$\bar{\phi} = \sum_{m=1}^{\infty} \bar{q}_m \frac{a^m \sin m\theta}{r^m} + \sum_{m=1}^{\infty} \frac{r^m}{m!}\sin m\theta \sum_{n=1}^{\infty} \frac{\bar{q}_n a^n}{(n-1)!}\frac{(n+m-1)!}{(2h)^{n+m}}, \tag{46}$$

where $\bar{q}_m$ is obtained from the following equation

$$-\frac{m}{a}\bar{q}_m + \sum_{n=1}^{\infty} \bar{q}_n \frac{a^{n+m-1}(n+m-1)!}{(m-1)!\,(n-1)!\,(2h)^{m+n}} = \delta_{m1}. \tag{47}$$

Substituting (46) into (45), we may write $\bar{\phi}$ as

$$\bar{\phi} = \sum_{m=1}^{\infty} \bar{q}_m \frac{a^m \sin m\theta}{r^m} + \sum_{m=1}^{\infty} \frac{r^m}{m a^{m-1}} \sin m\theta\left(\delta_{m1} + \frac{m}{a}\bar{q}_m\right), \tag{48}$$

from which, together with (6), we obtain

$$m_1 = \frac{\partial^2 \bar{\phi}}{\partial r\,\partial x} = \frac{1}{a^2}\sum_{m=1}^{\infty}\left[-m(m-1)\,\bar{q}_{m-1} + (m+1)\,m\bar{q}_{m+1}\right]\cos m\theta, \tag{49a}$$

$$m_3 = \frac{\partial^2 \bar{\phi}}{\partial r\,\partial z} = \frac{1}{a^2}\sum_{m=1}^{\infty}\left[m(m-1)\,\bar{q}_{m-1} + (m+1)\,m\bar{q}_{m+1}\right]\sin m\theta. \tag{49b}$$

These results now enable one to calculate the hydrodynamic coefficients τ_{ij}. Since the diagonal terms τ_{11} and τ_{33} for the submerged body are independent of forward speed (in this first-order theory), we only consider τ_{13} here (noticing $\tau_{13} = -\tau_{31}$). The integration over the body surface given in (34) is first evaluated as

$$\int_{S_0} (m_1\phi_3^0 - m_3\phi_1^0)\,\mathrm{d}S = -\frac{4\pi}{a}\sum_{m=2}^{\infty} m(m-1)\,\bar{q}_{m-1}p_m, \tag{50}$$

where we have used the relation $p_m = q_m$. To calculate the free-surface integral in (34), we write the potential on $z = 0$ as

$$\phi_3^0 = \sum_{m=1}^{\infty} p_m \frac{2a^m}{(m-1)!} ⨍_0^{\infty} \frac{k^m}{k-\nu}\mathrm{e}^{-kh}\cos kx\,\mathrm{d}k, \tag{51a}$$

$$\phi_1^0 = \sum_{m=1}^{\infty} q_m \frac{2a^m}{(m-1)!} ⨍_0^{\infty} \frac{k^m}{k-\nu}\mathrm{e}^{-kh}\sin kx\,\mathrm{d}k. \tag{51b}$$

Thus

$$\int_{S_F} [\phi_1^0 \phi_{3x}^0 - \phi_{1x}^0 \phi_3^0]\,\mathrm{d}x =$$

$$-2\int_0^\infty \sum_{m=1}^\infty \sum_{n=1}^\infty q_m p_n \frac{4a^{m+n}}{(m-1)!\,(n-1)!}[H_m(x)\,H_{n+1}(x) + H'_{m+1}(x)\,H'_n(x)]\,\mathrm{d}x, \quad (52)$$

where

$$\begin{aligned} H_m(x) &= \oint_0^\infty \frac{k^m}{k-\nu}\mathrm{e}^{-kh}\sin kx\,\mathrm{d}k \\ &= \frac{(m-1)!\sin m\theta_0}{r_0^m} + \frac{\nu(m-2)!\sin(m-1)\,\theta_0}{r_0^{m-1}} + \dots \\ &\quad + \nu^m\left[\pi\mathrm{e}^{-\nu h - \mathrm{i}\nu x} + \int_0^\infty \frac{\mathrm{e}^{-kx}(-k\sin kh - \nu\cos kh)}{k^2+\nu^2}\mathrm{d}k\right]; \end{aligned} \quad (53a)$$

$$\begin{aligned} H'_m(x) &= \oint_0^\infty \frac{k^m}{k-\nu}\mathrm{e}^{-kh}\cos kx\,\mathrm{d}k \\ &= \frac{(m-1)!\cos m\theta_0}{r_0^m} + \frac{\nu(m-2)!\cos(m-1)\,\theta_0}{r_0^{m-1}} + \dots \\ &\quad + \nu^m\left[-\pi\mathrm{i}\mathrm{e}^{-\nu h - \mathrm{i}\nu x} + \int_0^\infty \frac{\mathrm{e}^{-kx}(k\cos kh - \nu\sin kh)}{k^2+\nu^2}\mathrm{d}k\right]; \end{aligned} \quad (53b)$$

and

$$r_0 = (x^2+h^2)^{\frac{1}{2}}, \quad \theta_0 = \tan^{-1}\frac{x}{h}.$$

The last integral in (53) is obtained by changing the integration path in the complex plane (Lamb 1932). By substitution of (50) and (52) into (34) and noticing $a_{13} = b_{13} = 0$, τ_{13} can be easily found.

The application of the linear theory to the submerged circular cylinder requires $h \gg a$. It has been argued (Tuck 1965; Grue & Palm 1985) that $h \geqslant 4a$ may be necessary in general, depending on the other physical parameters. The present example, however, concerns the validation of the low-forward-speed approximation, by comparison with results evaluated by a more complete theory. We thus consider the case $h = 2a$, because this highlights the effects under investigation.

Figures 1(*a*) and 1(*b*) give a comparison of added mass and damping sway–heave coupling coefficients from the slow-forward-speed approximation, and from a general solution of the linearized potential theory. The latter has been obtained by using the coupled finite-element method (Wu & Eatock Taylor 1987). This represents the potential in an inner fluid domain surrounding the body surface in terms of finite-element shape functions, and combines this with a boundary-element representation of the potential in the outer region. The coefficients are zero in the case of $U = 0$, and in the low-forward-speed theory they are linearly proportional to U. They can therefore conveniently be expressed in terms of the non-dimensional coefficient $\tau_{13}/\rho\pi a^2\omega^2(\omega U/g)$. The added mass and damping coefficients non-dimensionalized in this way are plotted against the dimensionless wavenumber νa in figure 1. The results from the present calculation were obtained by truncating the series at $m = 5$, and the finite element solution used 12 elements surrounding the body surface. With the radius a equal to 1 m, the forward speed is taken as $U = 0.1$ m/s in the finite element

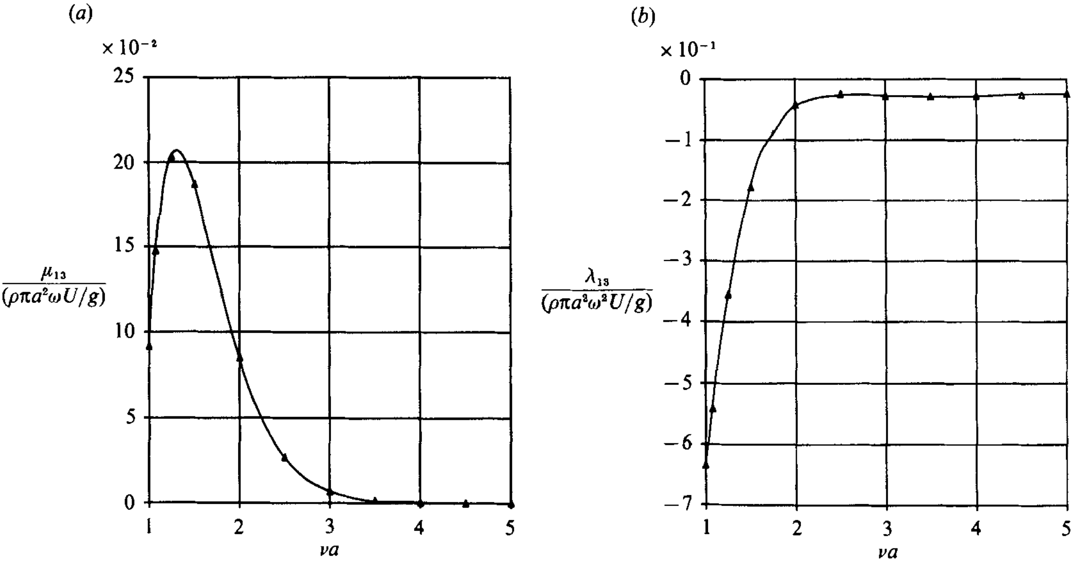

FIGURE 1. Sway–heave coupling coefficients for circular cylinder at low forward speed submerged at depth $h = 2a$: △, low-forward-speed approximation; ——, general numerical solution for $Fn = 0.03$. (*a*) Added mass coefficient; (*b*) damping coefficient.

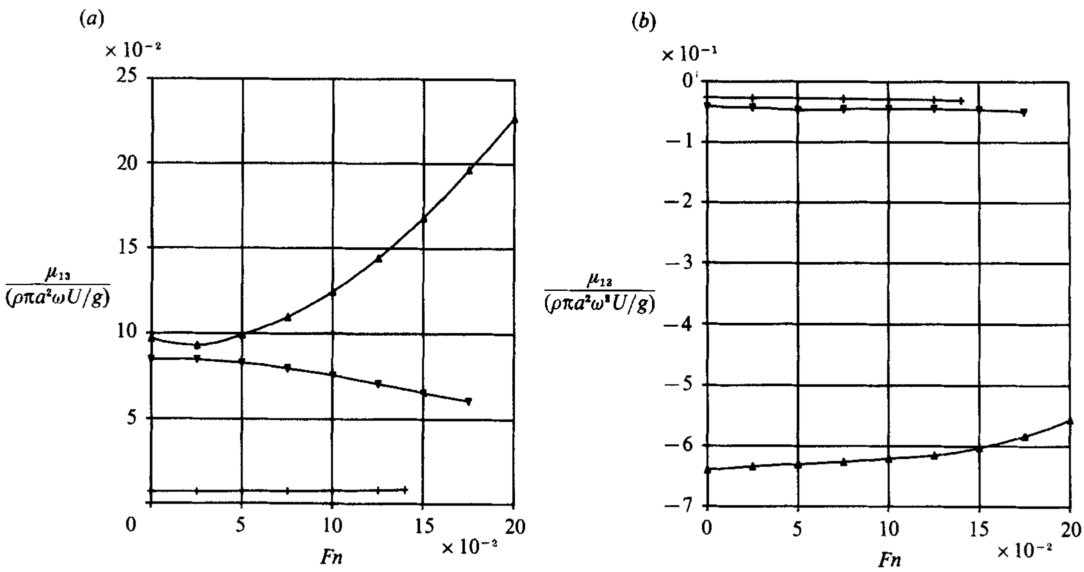

FIGURE 2. Variation of sway–heave coupling coefficients with forward speed, for a circular cylinder submerged at depth $h = 2a$: △, $\nu a = 1.0$; ▽, $\nu a = 2.0$; ×, $\nu a = 3.0$. (*a*) Added mass coefficient; (*b*) damping coefficient.

analysis, which corresponds to the Froude number $Fn = U/(ga)^{\frac{1}{2}} = 0.0319$. The figures show that the present low-forward-speed formulation provides a very accurate solution under these conditions. Furthermore, the successful comparison with results from the more complete procedure lends credibility to the low-forward-speed expansion, and to the comments made in the discussion following (27).

Figure 2 shows the variation of the hydrodynamic coefficients μ_{13} and λ_{13} with forward speed at different wavenumbers, within the range $U\omega/g < 0.25$. The limiting values at zero forward speed were obtained from the equations given above while the rest were evaluated by the coupled finite-element method. The results have been non-dimensionalized in a similar manner to those in figure 1. Thus according to the theory derived above they should be equal to a constant at a given wavenumber. It can be seen from the figures that the accuracy of the present results clearly depends on the relative magnitudes of Fn and νa. Since the low-forward-speed theory requires $Fn \ll (\nu a)^{\frac{1}{2}}$, it is expected that at lower frequency the applicability of the method is limited to a smaller range of Froude number. However it seems from the figures that the damping coefficients vary less with forward speed than do the added mass coefficients.

4. The nonlinear effect of the steady potential at the free surface

4.1. *Mathematical model*

As has been discussed, the linearized potential theory can usually only be applied to a slender or deeply submerged body. In general the free-surface condition for the steady and unsteady potentials should be written (Newman 1978) as in (7) and (8). The purpose of this section is to investigate the nonlinear coupling between steady and unsteady potentials at the free surface, as included in the first and third terms of (12). By including these additional terms we aim to derive a formulation more appropriate to a body at the free surface.

Once again we express the body motion potential ϕ_j as a perturbation expansion in the forward speed term, using (15). Substituting (15) into (12), we obtain the free-surface conditions on ϕ_j^0 and ϕ_j^1

$$\phi_{jz}^0 - \nu\phi_j^0 = 0, \tag{54}$$

$$\phi_{jz}^1 - \nu\phi_j^1 = 2\nu a(\bar{\phi}_x - 1)\,\phi_{jx}^0 + 2\nu a\bar{\phi}_y\,\phi_{jy}^0 - \nu a\phi_j^0\,\bar{\phi}_{zz}. \tag{55}$$

Comparing these two equations with (16) and (18), we can see that there are three additional terms in $\bar{\phi}$ in (55). However since $\bar{\phi} \to 0$ as $x^2 + y^2 \to \infty$, because of (9), (55) will reduce to (18) at infinity. Thus we can conclude that the nonlinear effect of the steady potential is localized if the terms in U^2 and higher are neglected. Correspondingly the radiation condition at infinity expressed by (27*b*) can be used here.

After a very similar derivation to that in the fully linearized problem, we obtain the equation for the hydrodynamic forces

$$\tau_{ij} = \tau_{ij}^{\mathrm{L}} - \rho i\omega U\nu \int_{S_F} (2\phi_i^0\,\phi_{jx}^0\,\bar{\phi}_x + 2\phi_i^0\,\phi_{jy}^0\,\bar{\phi}_y - \phi_i^0\,\phi_j^0\,\bar{\phi}_{zz})\,\mathrm{d}S, \tag{56a}$$

where superscript L indicates the contribution from the linearized theory. Alternatively, using $\bar{\phi}_{zz} = -\bar{\phi}_{xx} - \bar{\phi}_{yy}$ and Stokes theorem, we may write (56*a*) as

$$\tau_{ij} = \tau_{ij}^{\mathrm{L}} - \rho i\omega U\nu \left[\int_{S_F} (\phi_i^0\,\phi_{jx}^0\,\bar{\phi}_x - \phi_{ix}^0\,\phi_j^0\,\bar{\phi}_x + \phi_i^0\,\phi_{jy}^0\,\bar{\phi}_y - \phi_{iy}^0\,\phi_j^0\,\bar{\phi}_y)\,\mathrm{d}S - \int_{C_0} (\phi_i^0\,\phi_j^0\,\bar{\phi}_x\,\mathrm{d}y - \phi_i^0\,\phi_j^0\,\bar{\phi}_y\,\mathrm{d}x) \right]. \tag{56b}$$

But from the body-surface boundary condition (4) on the steady potential $\bar{\phi}$, we may also express the waterline integral as

$$\int_{C_0} \phi_i^0\,\phi_j^0(\bar{\phi}_x\,\mathrm{d}y - \bar{\phi}_y\,\mathrm{d}x) = \int_{C_0} \phi_i^0\,\phi_j^0\,\mathrm{d}y, \tag{57}$$

provided that the body intersects the free surface at a right angle. Hence the waterline integral in (56*b*) exactly cancels the waterline integral in (33), and as a result the relationship

$$\tau_{ij}(-U) = \tau_{ji}(U), \tag{58}$$

is always satisfied in this modified theory, irrespective of whether the body is slender or submerged. Furthermore, the diagonal terms τ_{jj} are seen now to be always independent of the forward speed, as long as terms in U^2 are neglected.

We may obtain the coefficients for the two-dimensional problem simply by omitting the derivatives with respect to y in (56*a*) and writing

$$\tau_{ij} = \tau_{ij}^{\mathrm{L}} - \rho\mathrm{i}\omega U\nu \int_{S_F} (2\phi_i^0\,\phi_{jx}^0\,\bar{\phi}_x - \phi_i^0\,\phi_j^0\,\bar{\phi}_{zz})\,\mathrm{d}S. \tag{59}$$

Proceeding as in the three-dimensional case, using Stokes theorem and the body-surface boundary condition on $\bar{\phi}$, we obtain

$$\tau_{ij} = \omega^2 a_{ij} - \mathrm{i}\omega b_{ij} + \rho\mathrm{i}\omega U \int_{S_0} (m_i\,\phi_j^0 - m_j\,\phi_i^0)\,\mathrm{d}S + \rho\mathrm{i}\omega U\nu\left[\int_{S_F} (\phi_i^0\,\phi_{jx}^0 - \phi_{ix}^0\,\phi_j^0)(1-\bar{\phi}_x)\,\mathrm{d}S\right]. \tag{60}$$

From this result we again find that (58) is satisfied, and τ_{jj} is independent of U, for any floating or submerged body.

4.2. *Results for a floating semicircular cylinder*

We consider the problem of floating semicircular cylinder of radius a in infinite water depth, as an example to demonstrate the nonlinear effect of the steady potential at the free surface on the hydrodynamic forces. Following Ursell's method (1949), we use multipole expansions of ϕ_j^0 in polar coordinates ($x = r\sin\theta$, $z = -r\cos\theta$)

$$\phi_3^0 = p_0\,\psi_0 + \sum_{m=1}^{\infty} p_m\,a^{2m}\left[\frac{\cos 2m\theta}{r^{2m}} + \frac{\nu}{2m-1}\,\frac{\cos(2m-1)\,\theta}{r^{2m-1}}\right], \tag{61a}$$

for heave and

$$\phi_1^0 = q_0\,\psi_1 + \sum_{m=1}^{\infty} q_m\,a^{2m+1}\left[\frac{\sin(2m+1)\,\theta}{r^{2m+1}} + \frac{\nu}{2m}\,\frac{\sin 2m\theta}{r^{2m}}\right], \tag{61b}$$

for sway, where

$$\psi_0 = \oint_0^\infty \frac{\mathrm{e}^{-kr\cos\theta}\cos(kr\sin\theta)}{k-\nu}\,\mathrm{d}k, \tag{62a}$$

and

$$\psi_1 = -a\frac{\partial\psi_0}{\partial x}. \tag{62b}$$

The coefficients p_m and q_m are given by the body-surface conditions

$$\frac{\partial\phi_3^0}{\partial r} = -\cos\theta, \tag{63a}$$

and

$$\frac{\partial\phi_1^0}{\partial r} = \sin\theta, \tag{63b}$$

when $r = a$. The solution of the infinite sets of (63*a*) and (63*b*) can be obtained by multiplying both sides by the complete series $\cos 2m\theta$ ($m = 0, 1, \ldots$) and $\sin 2m\theta$ ($m = 1, 2, \ldots$) respectively, and integrating the results from 0 to $\tfrac{1}{2}\pi$.

We have shown that for a floating body which intersects the free surface at a right angle the diagonal terms τ_{jj} from this theory are equal to those without forward speed, and since the cylinder is symmetric this will also be the case using the fully linear theory of §3. Thus we only discuss here the coupling terms τ_{13} and τ_{31}. We first use the linear theory to obtain τ_{13}^{L} and τ_{31}^{L}. From (32) we see that τ_{13}^{L} involves an integral over S_F and S_∞: this can be expressed in terms of

$$I(\nu a) = \int_a^{+\infty} \phi_1^0 \phi_{3x}^0 \,\mathrm{d}x - \nu \int_{S_{+\infty}} \phi_1^0 \phi_3^0 \,\mathrm{d}S. \tag{64}$$

Substituting (61) into (64), we obtain

$$I(\nu a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_m q_m (-1)^{m+n+1} \frac{2m}{2n+2m+1} + q_0 \sum_{m=1}^{\infty} 2m(-1)^{m+1} p_m L_{2m+1}$$
$$+ p_0 \sum_{m=1}^{\infty} q_m (-1)^{m+1} L_{2m+1} + p_0 q_o M, \tag{65}$$

where

$$L_{2m+1} = \frac{1}{2m+1} - \nu a \int_1^\infty \frac{f(\nu a x)}{x^{2m+1}} \mathrm{d}x + \pi \nu a \left\{ \frac{\mathrm{e}^{-\mathrm{i}\nu a}}{2m} + \frac{-\mathrm{i}\nu a\, \mathrm{e}^{-\mathrm{i}\nu a}}{2m(2m-1)} \right.$$
$$\left. + \ldots + \frac{(-\mathrm{i}\nu a)^{2m} \mathrm{e}^{-\mathrm{i}\nu a}}{(2m)!} + \frac{(-\mathrm{i}\nu a)^{2m}}{(2m)!} [-\mathrm{Ci}(\nu a) + \mathrm{i}\,\mathrm{si}\,(\nu a)] \right\}, \tag{66a}$$

$$M = -1 + 2\nu a \int_1^\infty \frac{f(\nu a x)}{x} \mathrm{d}x - (\nu a)^2 \int_1^\infty f^2(\nu a x) \,\mathrm{d}x$$
$$-2\nu a \pi [-\mathrm{Ci}\,(\nu a) + \mathrm{i}\,\mathrm{si}\,(\nu a)] + 2(\nu a)^2 \pi \int_1^\infty \mathrm{e}^{-\mathrm{i}\nu a x} f(\nu a x) \,\mathrm{d}x - \frac{\nu a \pi^2}{2\mathrm{i}} \mathrm{e}^{-2\mathrm{i}\nu a}. \tag{66b}$$

$\mathrm{Ci}\,(x)$ and $\mathrm{si}\,(x)$ are cosine and sine integrals (Abramowitz & Stegun 1965) and

$$f(x) = \mathrm{Ci}\,(x) \sin x - \mathrm{si}\,(x) \cos x. \tag{67}$$

The details of these derivations are given in the Appendix.

At low forward speed, the steady potential $\bar{\phi}$ for the semicircular cylinder can be obtained as

$$\bar{\phi} = -\frac{a^2}{r} \sin\theta. \tag{68}$$

From (6c), we obtain

$$m_1 = \frac{2\cos 2\theta}{a}, \tag{69a}$$

$$m_3 = \frac{2\sin 2\theta}{a}. \tag{69b}$$

Substituting (69) and (64) into (32), we obtain

$$\tau_{13}^{L} = 4\rho \mathrm{i}\omega U \int_0^{\frac{1}{2}\pi} [\cos 2\theta \phi_3^0 - \sin 2\theta \phi_1^0] \,\mathrm{d}\theta + 4\rho \mathrm{i}\omega U \nu I(\nu a). \tag{70}$$

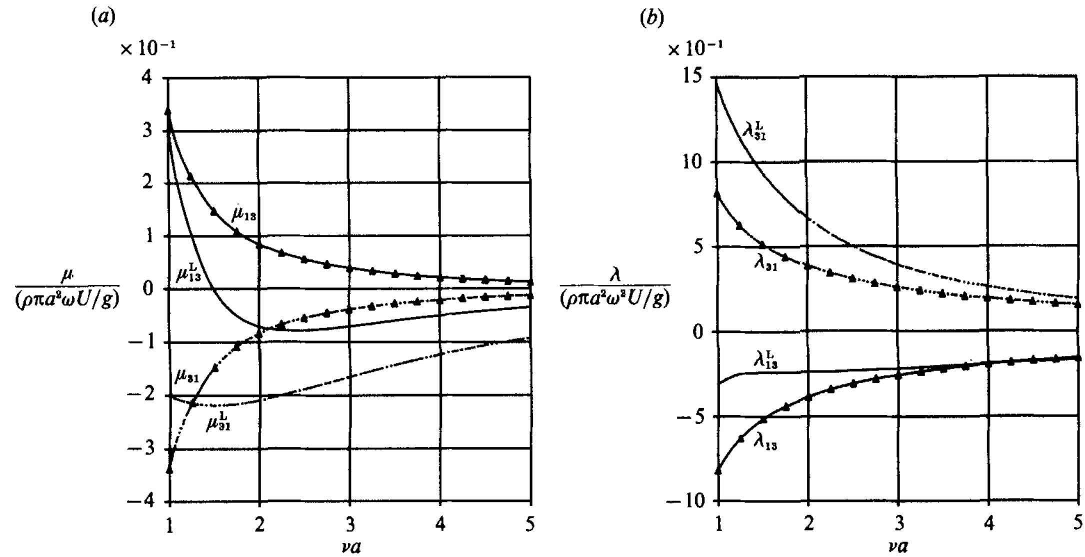

FIGURE 3. Sway–heave coupling coefficients for a floating semicircular cylinder based on low-forward-speed approximation. (a) Added mass coefficient; (b) damping coefficient.

Similarly, we have the integral over the free surface and at infinity for τ_{31}^{L}

$$\begin{aligned} H(\nu a) &= \int_a^{+\infty} \phi_{1x}^0 \phi_3^0 \,\mathrm{d}x - \nu \int_{S_{+\infty}} \phi_1^0 \phi_3^0 \,\mathrm{d}S \\ &= -\phi_1^0(a)\,\phi_3^0(a) - \int_a^{\infty} \phi_1^0 \phi_{3x}^0 \,\mathrm{d}x + \nu \int_{S_{+\infty}} \phi_1^0 \phi_3^0 \,\mathrm{d}S \\ &= -\phi_1^0(a)\,\phi_3^0(a) - I(\nu a). \end{aligned} \tag{71}$$

From (32), we obtain

$$\begin{aligned} \tau_{31}^{\mathrm{L}} &= \rho\mathrm{i}\omega U \int_{S_0} (m_3 \phi_1^0 - m_1 \phi_3^0)\,\mathrm{d}S + 4\rho\mathrm{i}\omega U\nu H(\nu a) \\ &= -\rho\mathrm{i}\omega U \int_{S_0} (m_1 \phi_3^0 - m_3 \phi_1^0)\,\mathrm{d}S - 4\rho\mathrm{i}\omega U\nu I(\nu a) - 4\rho\mathrm{i}\omega U\nu\phi_1^0(a)\,\phi_3^0(a) \\ &= -\tau_{13}^{\mathrm{L}} - 4\nu\rho\mathrm{i}\omega U\phi_1^0(a)\,\phi_3^0(a). \end{aligned} \tag{72}$$

Following a very similar derivation, we can obtain the equations for τ_{13} and τ_{31} including the nonlinear effect of the steady potential at the free surface. From (60), (68) and (69) we obtain

$$\tau_{13} = 4\rho\mathrm{i}\omega U \int_0^{\frac{1}{2}\pi} [\cos 2\theta\phi_3^0 - \sin 2\theta\phi_1^0]\,\mathrm{d}\theta + 2\rho\mathrm{i}\omega U\nu K(\nu a), \tag{73}$$

where

$$K(\nu a) = \int_a^{\infty} (\phi_1^0 \phi_{3x}^0 - \phi_{1x}^0 \phi_3)(1 - \bar{\phi}_x)\,\mathrm{d}x. \tag{74}$$

The evaluation of $K(\nu a)$ is similar to that of $I(\nu a)$.

Values of τ_{13} and τ_{31} for the semicircular cylinder are presented in figures 3(a) and 3(b) as added mass and damping coefficients, non-dimensionalized by $(\omega U/g)\rho\pi a^2\omega^2$, and plotted against dimensionless wavenumber νa. It can be seen from the figures that the nonlinear effect of the steady potential is important in the range of frequency plotted. It seems, however, that the nonlinear influence on the damping coefficients becomes less significant as the frequency increases. It may also be noticed that the solution from the linear theory does not satisfy the reverse flow relationship because of the 'line integral effect'; but inclusion of the nonlinear coupling with the steady potential eliminates this difference.

5. Concluding remarks

The present formulations are part of an investigation into the behaviour of oscillating bodies at low forward speed. The foregoing analysis has shown that hydrodynamic forces may be effectively calculated using the low speed theory. In addition to the assumptions of ideal fluid flow, the theory is limited by the following conditions: the oscillatory body motions are small; the Froude number $U/(gl)^{\frac{1}{2}} \ll 1$, where l is a characteristic lengthscale in the steady forward speed problem; $U/\omega a \ll 1$ and $U\omega/g < 0.25$.

These assumptions lead to the perturbation theory based on a small forward speed parameter. Although the resulting perturbation expansion for the velocity potential is not uniformly valid as $R \rightarrow \infty$, we argue that the approach is justified by the foregoing restrictions and because we only use the resulting series to obtain integrated forces on the ship. Where we have made comparisons with the results of a more complete theory (for the more severe test of a two-dimensional problem), these have fully confirmed the validity of the approach.

The fully linear theory developed in §3 is based on specification of a boundary-value problem which has received considerable attention previously. It is applicable to deeply submerged bodies, or slender bodies at the free surface. The low-speed analysis suggests that in this form the theory leads to hydrodynamic coefficients which in principle may not satisfy the reverse flow relationships proposed by Timman & Newman (1962). It can be argued however that consistency is retained, since these relations are found to be satisfied exactly for submerged bodies; and for slender bodies at the free surface they are approximately satisfied, since the waterline integral in (33) is then negligible.

By including additional coupling effects between the steady and unsteady potentials in the free-surface boundary condition, we have introduced a degree of nonlinearity into the conventional formulation. The resulting theory has been applied to a body at the free surface, and it has been found that the reverse flow relationships are now satisfied, irrespective of whether the body is slender or not.

These effects have been illustrated by obtaining analytical solutions for a submerged circular cylinder and a floating semicircular cylinder. In principle the same general formulations based on the low-speed approximation may be applied to arbitrary bodies in two or three dimensions. The major difficulty is the evaluation of the free-surface integral, but this may be overcome by adopting procedures similar to those used in calculation of the second-order diffraction force on a body in waves (e.g. Eatock Taylor & Hung 1987). The results obtained here give an indication of the importance of the forward speed effect for cylinders, even at low speeds, and also illustrate the importance of the nonlinear effect due to the steady potential at the free surface.

This work was partially supported by the joint SERC/MOD grant XG10107 to the London Centre for Marine Technology. The authors would also like to thank Mr C. S. Hu for some useful discussions.

Appendix. Evaluation of the term $I(\nu a)$ in equation (64)

The integration over S_F and S_∞ in (32) for a circular cylinder can be written as

$$I(\nu a) = \int_a^\infty \left\{ p_0 \frac{\partial \psi_0}{\partial x} + \sum_{m=1}^{\infty} p_m a^{2m} \left[-\frac{2m}{x^{2m+1}} \cos m\pi \right] \right\}$$
$$\times \left\{ q_0 \psi_1 + \sum_{m=1}^{\infty} q_m a^{2m+1} \left[\frac{1}{x^{2m+1}} \sin (2m+1) \tfrac{1}{2}\pi \right] \right\} dx + I_{+\infty}, \quad (A\ 1)$$

where $I_{+\infty}$ is the integral over $S_{+\infty}$ given by

$$I_{+\infty} = -\nu \int_{-\infty}^{0} \psi_0 \psi_1 \, \mathrm{d}z \quad (x \to +\infty).$$

Using (62*b*), (A 1) becomes

$$I(\nu a) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_m q_n (-1)^{m+n+1} \frac{2m}{2m+2n+1} + q_0 \sum_{m=1}^{\infty} 2m(-1)^{m+1} p_m L_{2m+1}$$
$$+ p_0 \sum_{m=1}^{\infty} q_m (-1)^{m+1} L_{2m+1} + p_0 q_0 M, \quad (A\ 2)$$

where

$$L_{2m+1} = a^{2m} \int_a^\infty \frac{\psi_1}{x^{2m+1}} \mathrm{d}x, \quad (A\ 3)$$

$$M = -\frac{1}{a} \int_a^\infty \psi_1^2 \, \mathrm{d}x + I_{+\infty}. \quad (A\ 4)$$

To compute L_{2m+1} and M, we first write ψ_1 on the free surface

$$\psi_1 = \frac{a}{x} + \nu a \mathrm{PV} \int_0^\infty \frac{\sin (kx)}{k-\nu} \mathrm{d}k - \nu a \pi \mathrm{i} \sin (\nu x)$$
$$= \frac{a}{x} + \nu a \mathrm{PV} \int_0^\infty \frac{\sin (k)}{k-\nu x} \mathrm{d}k - \nu a \pi \mathrm{i} \sin (\nu x)$$
$$= \frac{a}{x} + \nu a \mathrm{PV} \int_{-\nu x}^\infty \frac{\sin k \cos (\nu x) + \cos k \sin (\nu x)}{k} \mathrm{d}k - \nu a \pi \mathrm{i} \sin (\nu x)$$
$$= \frac{a}{x} + \nu a \left\{ \cos \nu x \left[\int_0^{\nu x} \frac{\sin k}{k} \mathrm{d}k + \tfrac{1}{2}\pi \right] + \sin \nu x \int_{\nu x}^\infty \frac{\cos k}{k} \mathrm{d}k \right\} - \nu a \pi \mathrm{i} \sin (\nu x)$$
$$= \frac{a}{x} + \nu a \cos (\nu x) [\mathrm{Si}(\nu x) + \tfrac{1}{2}\pi] - \nu a \sin \nu x \, \mathrm{Ci}(\nu x) - \nu a \pi \mathrm{i} \sin (\nu x)$$
$$= \frac{a}{x} - \nu a f(\nu x) + \nu a \pi \, \mathrm{e}^{-\mathrm{i}\nu x} \quad (x > 0), \quad (A\ 5)$$

where Ci (x) and Si (x) are cosine and sine integrals respectively and

$$\mathrm{Si}\,(x) = \mathrm{si}\,(x) + \tfrac{1}{2}\pi, \tag{A 6}$$

$$f(x) = \mathrm{Ci}\,(x)\sin x - \mathrm{si}\,(x)\cos x, \tag{A 7}$$

as defined by Abramowitz & Stegun (1965). Thus

$$\begin{aligned} L_{2m+1} &= \frac{1}{2m+1} - \nu a \int_1^\infty \frac{f(\nu a x)}{x^{2m+1}}\,\mathrm{d}x + \nu a \pi \int_1^\infty \frac{\mathrm{e}^{-\mathrm{i}\nu a x}}{x^{2m+1}}\,\mathrm{d}x \\ &= \frac{1}{2m+1} - \nu a \int_1^\infty \frac{f(\nu a x)}{x^{2m+1}}\,\mathrm{d}x + \pi \nu a \left\{ \frac{\mathrm{e}^{-\mathrm{i}\nu a}}{2m} + \frac{-\mathrm{i}\nu a\,\mathrm{e}^{-\mathrm{i}\nu a}}{2m(2m-1)} \right. \\ &\quad \left. + \ldots + \frac{(-\mathrm{i}\nu a)^{2m}\,\mathrm{e}^{-\mathrm{i}\nu a}}{(2m)!} + \frac{(-\mathrm{i}\nu a)^{2m}}{(2m)!}[-\mathrm{Ci}\,(\nu a) + \mathrm{i}\,\mathrm{si}\,(\nu a)] \right\}. \end{aligned} \tag{A 8}$$

Similarly, we have

$$\begin{aligned} M &= -\frac{1}{a}\int_a^\infty \left[\frac{a}{x} + \nu a \pi\,\mathrm{e}^{-\mathrm{i}\nu x} - \nu a f(\nu x)\right]^2 \mathrm{d}x + I_{+\infty} \\ &= -\int_1^\infty \left\{\left[\frac{1}{x} - \nu a f(\nu a x)\right]^2 + 2\nu a \pi\,\mathrm{e}^{-\mathrm{i}\nu a x}\left[\frac{1}{x} - \nu a f(\nu a x)\right]\right\}\mathrm{d}x + \frac{(\nu a \pi)^2}{2\mathrm{i}\nu a}\,\mathrm{e}^{-2\mathrm{i}\nu a x}\Big|_1^\infty + I_{+\infty}. \end{aligned}$$

Noticing the contributions from infinity will cancel each other, we have

$$\begin{aligned} M = -1 + 2\nu a \int_1^\infty \frac{f(\nu a x)}{x}\,\mathrm{d}x - (\nu a)^2 \int_1^\infty f^2(\nu a x)\,\mathrm{d}x - 2\nu a \pi[-\mathrm{Ci}\,(\nu a) + \mathrm{i}\,\mathrm{si}\,(\nu a)] \\ + 2(\nu a)^2 \pi \int_1^\infty \mathrm{e}^{-\mathrm{i}\nu a x} f(\nu a x)\,\mathrm{d}x - \frac{\nu a \pi^2}{2\mathrm{i}}\,\mathrm{e}^{-2\mathrm{i}\nu a}. \end{aligned} \tag{A 9}$$

REFERENCES

Abramowitz, M. & Stegun, M. 1965 *Handbook of Mathematical Functions*. Dover.

Chang, M. S. 1977 Computations of three-dimensional ship-motions with forward speed. *Second Intl Conf. on Num. Ship Hydrodyn.*, pp. 124–135.

Eatock Taylor, R. & Hung, S. M. 1987 Second order diffraction forces on a vertical cylinder in regular waves. *Appl. Ocean Res.* **9**, 19–30.

Grue, J. & Palm, E. 1985 Wave radiation and wave diffraction from a submerged body in a uniform current. *J. Fluid Mech.* **151**, 257–278.

Guevel, P. & Bougis, J. 1982 Ship-motions with forward speed in infinite depth. *Intl Shipbuilding Prog.* **29**, 103–117.

Haskind, M. D. 1946 The hydrodynamic theory of ship oscillations in rolling and pitching. *Prikl. Math. Mech.* **10**, 33–66.

Huijsmans, R. H. M. 1986 Wave drift forces in current. *16th Conf. on Naval Hydrodyn.*

Inglis, R. B. & Price, W. G. 1981 A three dimensional ship motion theory – comparison between theoretical predictions and experimental data of the hydrodynamic coefficients with forward speed. *Trans. RINA* **124**, 141–157.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Mei, C. C. 1982 *The Applied Dynamics of Ocean Surface Waves*. Wiley-Interscience.

Newman, J. N. 1965 The exciting forces on a moving body in waves. *J. Ship Res.* **9**, 190–199.

Newman, J. N. 1978 The theory of ship motions. *Adv. Appl. Mech.* **18**, 221–283.

Ogilvie, T. F. 1963 First- and second-order forces on a cylinder submerged under a free surface. *J. Fluid Mech.* **16**, 451–472.

OGILVIE, T. F. & TUCK, E. O. 1969 A rational strip theory for ship motions: part 1. *Dept. Nav. Archit. Mar. Engng Rep. no.* 013. University of Michigan.

THORNE, R. C. 1953 Multipole expansions in the theory of surface waves. *Proc. Camb. Phil. Soc.* **49**, 707–716.

TIMMAN, R. & NEWMAN, J. N. 1962 The coupled damping coefficients of a symmetric ship. *J. Ship Res.* **5**, 1–7.

TUCK, E. O. 1965 The effect of non-linearity at the free surface on flow past a submerged cylinder. *J. Fluid Mech.* **22**, 401–414.

URSELL, F. 1949 On the heaving motion of a circular cylinder on the surface of a fluid. *Q. J. Mech. Appl. Maths* **2**, 218–231.

WU, G. X. & EATOCK TAYLOR, R. 1987 Hydrodynamic forces on submerged oscillating cylinders at forward speed. *Proc. R. Soc. Lond.* A **414**, 149–170.

WU, G. X. & EATOCK TAYLOR, R. 1988 Reciprocity relations for hydrodynamic coefficients of bodies with forward speed. *Intl Shipbuilding Prog.* **35**, 145–153.